PID control under sampling period constrains

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Abstract. Digital redesign by emulation of the continuous is a quite frequent method for building digital controllers. However, as the sampling period increases the performance is degraded. One solution is to design the controller directly in the discrete domain, with the advantage of having the sampling period as an extra parameter. But when there are sampling constrains either in sensors or actuators it is necessary to choose as sampling period the slowest one. Moreover, such a period becomes a lower bound. Multi-rate controllers arise due to these sampling constrains releasing from the hypothesis of having a single period. This is why it is interesting to consider a so widespread controller such as the PID in multi-rate situations. In this paper a direct design method, based on Modified Ziegler Nichols, for PID controllers is presented. The novelty of this paper is to consider the sampling period as a degree of freedom and, when it comes too large, to improve the performance by turning it into a multi-rate PID. With the method proposed both gain and phase margin requirements can be obtained.

Key words: Multi-rate, PID, Ziegler-Nichols, Tuning methods, Kranc operators

1 Introduction

It is remarkable that the design of controllers is still carried out in the continuous domain even when these are finally built in digital devices. Just considering PID controllers, there are about one thousand tuning rules compiled in [1], all of them for continuous systems. Afterwards, taking a sampling period fast enough and then applying some approximation of the derivative such as Backward differences or Tustin, a discrete PID is obtained. However this method is no longer right if the sampling period is above a threshold. Designing techniques directly in the discrete domain considering the sampling period as an extra degree of freedom have been proposed [2],[3] . Besides sampling constrains either on sensors or actuators can prevent from using a single sampling rate or even from choosing it within a given range, let alone small enough to emulate the continuous. A system under this constrains is said to be Multi-rate (MR). There are two possible control strategies: Multi-Rate Input Controller (MRIC) in which the manipulated

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variable is updated faster than the measured variable, and Multi-Rate Output Controller (MROC) which is the complementary case. Chemical processes [4], polymer reactors [5] or sector servo of hard disk drives [6] are MRIC examples where the control action could be updated N_u times for each measure of the output. On the other hand, artificial vision systems are a MROC example in which the large amount of incoming data is faster than the algorithm working on it.

Multi-rate PID (MRPID) controllers have been built for both strategies with different techniques [7]-[12]. All prove very good performance but the controller obtained is not expressed in term of the classical gains K_P, K_I and K_D. Using Kranc operators technique [13] the authors presented a model for a MRIC PID controller that preserves the gains [14] but has the drawback of dealing with MIMO systems.

This work considers the general case for MRPID in which the measured variable is updated every T_e seconds and the manipulated variable every T_u seconds. As a consequence both MRIC and MROC strategies are covered. The contribution of this paper is to present the advantages of such a MRPID for obtaining a range of useful phase margins and a given gain margin, bringing it face to face with the conventional single-rate PID (SRPID) in two different scenarios: first taking the sampling period as a parameter design; and then assuming there is a lower bound for the sampling period which is the slowest one found in the control system. The disadvantages remarked above are coped so that finally a simple tuning method in the frequency domain is obtained. The rest of the paper is organised as follows. Section II presents the MRPID controller and its modelling. Section III is devoted to the tuning method in the frequency domain. Section IV shows an study case and finally the conclusions are given.

Notation

$G(s)$:	Continuous transfer function
$G(z), G^{T_s}(z)$:	Discrete transfer function (from an algorithm with period T_s)
$\underline{G}(z)$:	Kranc Operator of G
l.c.m.$(a, b, c, ...)$:	Least common multiple of a, b, c, ...
$\underline{R}^T, \underline{E}^T, \underline{U}^T, \underline{Y}^T$:	Discrete transfer function of vectorised signals sampled at metaperiod T.

N_e :	Error rate	N_u :	Control rate
N_P :	Proportional rate	N_I :	Integral rate
N_D :	Derivative rate	T :	Metaperiod
m:	$l.c.m.(N_e, N_P, N_I, N_D)$	n:	$l.c.m.(N_u, N_P, N_I, N_D)$

2 Model

Let us consider a parallel digital PID controller in which the backward differences approximation of the derivative, $s = (z-1)/zT$, is used. Let us assume that the error is measured with period T_e and the control action is updated with period T_u. Hence, there are two intuitive approaches to the way the controller internally works. The simplest one is to compute the control action value in terms of the last error sample and send it out with period T_u until a new error sample is received. By doing so a SRPID working at the larger period is finally obtained. A more elaborate and natural decision is to consider the error as a sequence of steps and the control action as the sum of every basic control component (i.e. proportional, integral and derivative) response to such steps. Still a more general case can be considered with this approach if the proportional, integral and derivative components work at periods T_{prp}, T_{int} and T_{der} respectively. This scheme is depicted in Fig. 1(a), where the superscript gives extra information about the sampling time to the Z-transform.

Any given basic action will seldom take place at the same instant than any other one or even the final control action or the error sampling. Therefore, following the second approach, the simplest solution to this problem is to hold the error sample for T_e seconds so that it is always available for every component. Every basic action is also hold during its own period making feasible to add the three of them at any time. This output is finally sampled every T_u seconds. Kranc operators methodology [13] has been used throughout this paper. Kranc operators transform a MR system into a Single Rate MIMO system working at metaperiod T, defined as the l.c.m. of all the periods found in the system. The two basic ideas behind Kranc operators are the vector switch decomposition (VSD) [15] and the approximation of continuous signals by fictitious samplers at high rates [16]. Thus a sampler element with period T/N is modelled by an expand operator $[N^+]$ which vectorises the input signal in N channels so that every channel is advanced with respect to the next one T/N seconds; then all the channels are sampled at metaperiod T so that in T seconds N samples of the signal are taken and finally the signal is reconstructed with a reduce operator $[N^-]$ as defined in [13]. So, first of all, the periods appearing in Fig. 1(a) must be rewritten in terms of the metaperiod T: $T_e = T/N_e$, $Tu = T/N_u$, $T_{prp}=T/N_P$, $T_{int}=T/N_I$, $T_{der}=T/N_D$. Due to space limitations and since of the paper is focused on the tuning method in the frequency domain we encourage to find an exhaustive description of the model in [17]

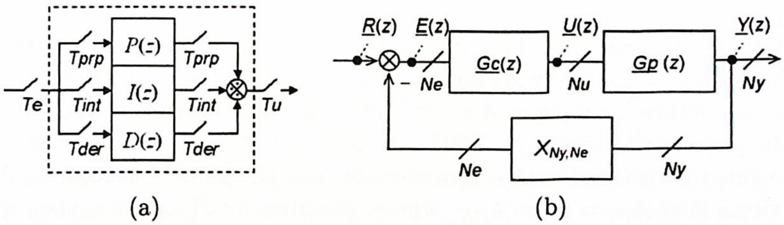


Fig. 1. MRPID scheme (a) and block diagram using Kranc Operators (b).

When the MRPID controller is inserted in a control loop, the process and the feedback must be also modelled using Kranc operators [13]. The control loop is finally represented by the block diagram shown in Fig. 1(b) Here $Gc(z)$ is the model of the MRPID controller obtained above and $Gp(z)$ is the Kranc operator of the continuous process $Gp(s)$ preceded by a ZOH with period T/N_u. The continuous output of the process is approximated with a fictitious sampler that measures N_y times in a metaperiod. Since just N_e out of these N_y samples must be fed back, the block $X_{Ny,Ne}$ is necessary. The figure also represents the vectorisation by a crossing line that indicates the number of channels it has in.

3 Tuning Method

The open loop transfer matrix of the MR system shown in Fig. 1 is given by:

$$M(z) = X_{Ny,Ne} \cdot \underline{Gp}(z)\underline{Gc}(z), \tag{1}$$

Thus $M(z)$ is a square transfer matrix with N_e rows and columns and therefore the Nyquist curve must be generalised to the characteristic loci [18], which is the eigenvalue set of $M(e^{j\omega T})$ for $0 \leq \omega \leq \pi/T$. In order to generalise the Modified ZN method to the MR case, first note that

$$\underline{XG}(z) = X_{Ny,Ne} \cdot \underline{Gp}(z) = [g_{ij}(z)], \tag{2}$$

with $1 \leq i \leq Ne$ and $1 \leq j \leq Nu$, is the open loop transfer matrix without the controller. Let us consider that a MR Proportional controller $\underline{C_P}(z)$ with $K_P = 1$ and $N_P = 1$ is connected. Such a controller is represented by a transfer matrix in which every element is equal to zero but the first column is equal to one. Then the element in the position (i,j) of $M_P(z) = \underline{XG}(z)\underline{C_P}(z)$ is given by:

$$m_{ij} = \begin{cases} \sum_{q=1}^{Nu} g_{i,q}(z) & \text{if } j = 1 \\ 0 & \text{if } 1 < j \leq Ne \end{cases} \tag{3}$$

The eigenvalue set of the matrix $M_P(e^{j\omega T})$ is

$$\lambda_{1,2,\ldots,Ne} = \sum_{q=1}^{Nu} g_{1,q}(e^{j\omega T}) = Gp^T(e^{j\omega T}) \tag{4}$$

with multiplicity N_e. This means that inserting a Proportional controller with $K_P = N_P = 1$, for any N_e and N_u, into the control loop and selecting a frequency ω_0 we determine a point $A = Gp^T(e^{j\omega_0 T})$, which belongs to the Nyquist curve of the process preceded by a ZOH with period T. Moreover, since the Nyquist curve in this case is equal to the characteristic loci, the point A is actually N_e coincident points $A = A_{P1} = \ldots = A_{PN_e}$ where the subscript P indicates they were obtained just with a proportional controller. These A_{Pi} are complex numbers which have a well-known correspondence in $\mathbb{R}^2$, so from now on complex points will be referred to as elements (or points) of $\mathbb{R}^2$. Increasing the proportional internal rate N_P splits point A so that in the most general case $A \neq A_{P1} \neq ... \neq A_{PN_e}$. Then, modifying K_P moves every A_{Pi} over the line that goes through the origin and A_{Pi} so that $|A_{Pi}|$ is a function of K_P. Let $\underline{C_I}(z)$ be a MR Integral controller with $N_I = 1$ and $K_I = 0$, which is now connected instead of $\underline{C_P}(z)$ in (1). Following the same reasoning as above, point A is now split, by means of the controller, for the frequency ω_0, into N_e points $A_{I1} \neq ... \neq A_{IN_e} \neq A$; and again for each A_{Ii} its module $|A_{Ii}|$ is a function of K_I. The same can be said for a MR Derivative controller $\underline{C_D}(z)$ with $N_D = 1$ and $K_D = 0$ so that in the end each $|A_{Di}|$ is a function of K_D.

We are now in a position of considering the complete MRPID $\underline{Gc}(z) = \underline{C_P}(z) + \underline{C_I}(z) + \underline{C_D}(z)$, for any set of internal rates and gains. Once it is connected into the open loop, (1) becomes

$$M(z) = X_{Ny,Ne} \cdot \underline{Gp}(z)\left(\underline{C}_P(z) + \underline{C}_I(z) + \underline{C}_D(z)\right) \tag{5}$$

The eigenvalue set of (5) for $z = e^{j\omega_0 T}$ is therefore given by

$$\{B_i = A_{Pi} + A_{Ii} + A_{Di}\}, \text{ with } i = 1, ..., Ne \tag{6}$$

Consequently once the internal rates N_P, N_I and N_D as well as the frequency ω_0 have been fixed, the eigenvalue set of (5) depends only on K_P, K_I and K_D. In this paper we propose to choose gains K_P, K_I and K_D to assure a closed loop gain margin of γ dB, which means that the characteristic loci has satisfy two conditions: it must cut the negative x-axis in $B = -10^{-\gamma/20}$ and nowhere else to the left. This is the task the next two subsections are devoted to.

3.1 Computing the Proportional, Integral and Derivative main directions

Since N_e points B_i are obtained with (6) it is necessary to select just one. Let $B_1 = A_{P1} + A_{I1} + A_{D1}$ be the eigenvalue that will match the required point B, then we define the unitary vectors

$$\mathbf{V_P} = \frac{A_{P1}}{|A_{P1}|}, \ \mathbf{V_I} = \frac{A_{I1}}{|A_{I1}|}, \ \mathbf{V_D} = \frac{A_{D1}}{|A_{D1}|} \tag{7}$$

and the proportional, integral and derivative main directions as the directions established by $\mathbf{V_P}$, $\mathbf{V_I}$ and $\mathbf{V_D}$ respectively. It is clear that points A_{P1}, A_{I1} and A_{D1} must be properly chosen out of their respective sets $\{A_{Pi}\}$, $\{A_{Ii}\}$ and $\{A_{Di}\}$ for satisfying the conditions of the given gain margin. As a first approach we recommend to choose A_{P1} as

$$A_{P1} = \min_{A_{Pi}} \text{abs}\left(\text{angle}\left(\frac{A_{Pi}}{|A_{Pi}|}; Ne\right) - \text{angle}\left(\frac{A_P}{|A_P|}; 1\right)\right) \tag{8}$$

where the first term is the angle of the unitary vector established by A_{Pi} using the actual controller (i.e. with the given $N_e \geq 1$) and the second term is the angle of the unitary vector established by the unique A_P if N_e was 1. A similar criterion should be used for A_{I1} and A_{D1}.

3.2 Computing K_P, K_I and K_D

Now it is possible to generalise Modified ZN method to the MR situation shown in Fig. 1 with the following procedure:

1. Connect a MRPID controller with $N_P = N_I = N_D = K_P = 1$ and $K_I = K_D = 0$.
2. Select a frequency $\omega_0 \in [0, \omega_0/T)$. This frequency determines a point $A = Gp^T(e^{j\omega_0 T})$ in the Nyquist plot of the process preceded by a ZOH with period T .
3. Make N_P, N_I and N_D equal to the greater of N_e and N_u.
4. Obtain $\{A_{Pi}\}$, the eigenvalue set of the open loop transfer matrix considering only proportional action with $K_P = 1$. Similarly, obtain $\{A_{Ii}\}$ and $\{A_{Di}\}$ considering only integral ($K_I = 1$) and derivative action ($K_D = 1$) respectively.
5. Select $\{A_{P1}\}$, $\{A_{I1}\}$ and $\{A_{D1}\}$ according to (8) and obtain $\mathbf{V}_\mathrm{P}$, $\mathbf{V}_\mathrm{I}$ and $\mathbf{V}_\mathrm{D}$ with (7).
6. Select a gain margin of γ dB. This gain margin determines a point $B = -10^{-\gamma/20}$.
7. Point B can be rewritten as $B = \rho_P \mathbf{V}_\mathrm{P} + \rho_I \mathbf{V}_\mathrm{I} + \rho_D \mathbf{V}_\mathrm{D}$, where ρ_P, ρ_I and ρ_D depend on K_P, K_I and K_D respectively. Although the relation between K and ρ is hard (or impossible) to determine, an iterative and simple Newton algorithm rapidly converges on a solution. The main steps are:
   - Consider only proportional action and give an initial value to K_P.
   - Obtain $\{A_{Pi}\}$ for that K_P and select A_{P1} as the one in the direction of $\mathbf{V}_\mathrm{P}$.
   - If $|A_{P1}| = \rho_P$ then K_P is the solution. Otherwise repeat the process with another value of K_P.
   - Repeat the steps for integral and derivative actions, obtaining K_I and K_D.

It is important to remark the following issues:

1. Unlike for the continuous controller, $\mathbf{V}_\mathrm{P}$, $\mathbf{V}_\mathrm{I}$ and $\mathbf{V}_\mathrm{D}$ are generally not orthogonal.
2. The choice of $\mathbf{V}_\mathrm{P}$, $\mathbf{V}_\mathrm{I}$ and $\mathbf{V}_\mathrm{D}$ according to criterion (8) is not always right so, once the controller has been tuned, it is necessary to verify that point B is actually in the characteristic loci. If not then other main directions must be chosen.
3. The gain margin must be also checked out, for the method only assures one value of the characteristic loci which is point B.

3.3 Derivative action effect on both gain and phase margins

When using MRPI or MRPD controllers there is a unique couple (ρ_P, ρ_I) or (ρ_P, ρ_D) but for MRPID infinite combinations of ρ_P, ρ_I and ρ_D are possible so an extra constrain is needed for fixing a unique combination (ρ_P, ρ_I, ρ_D). When

$N_e = 1$, it has been proposed to try different K_P (i.e. ρ_P) for it has the effect of modifying the phase margin keeping the gain margin [14]. The same can be applied now for $N_e > 1$. This paper gets deeper in the study of when both gain and phase margins requirements can be satisfied. So far a frequency ω_0 must be previously fixed so that point A is moved to position B to satisfy the gain margin requirement; but another frequency could also accomplish the same goal, perhaps with a wider or more appropriate phase margin range. Thus, if ω_0 is a degree of freedom the following algorithm is proposed:

1. Carry out the procedure of computing K_P, K_I and K_D selecting the ultimate frequency as the initial value for ω_0 and fixing $K_D = 0$. Consequently the proportional gain gets its maximum value K_{Pmax}.
2. Fix $K_P = 0.95 K_{Pmax}$ and compute K_I and K_D again.
3. Repeat step 2 reducing another 5% each time until $K_P = 0$ or the gain margin is not satisfied because of the characteristic loci has been excessively bent. Thus a phase margin range has been obtained for the same gain margin.
4. Set another frequency ω_0, smaller than the previous one since it was the upper bound and repeat steps 1, 2 and 3. Thus a new phase margin range is obtained, probably intersecting with others so eventually a gain margin range for a given phase margin is also obtained

4 Example

Let us consider the continuous process given by the transfer function $Gp(s) = 25(s+1)^{-1}(s^2+s+25)^{-1}$ and a the following robustness requirements: a gain margin of $\gamma = 26$ dB and a phase margin $\phi = 55°$. Two controllers were tested under three error sampling periods: $T_e = \{50\text{ ms}, 125\text{ ms}, 250\text{ ms}\}$; the first one was a conventional discrete PID (SRPID) that updates the control action at $T_u = T_e$, and the second one a MRPID decreasing T_u to 40% of T_e; i.e. $T_u = \{20\text{ ms}, 50\text{ ms}, 100\text{ ms}\}$. Notice that both cases can be modelled by Kranc Operators technique, taking a metaperiod $T = \{100\text{ ms}, 250\text{ ms}, 500\text{ ms}\}$, with $N_e = N_u = N_P = N_I = N_D = 2$ for the SRPID and $Ne = 2; N_u = N_P = N_I = N_D = 5$ for the MRPID. Let $Gp^T(z)$ denote the discrete model of the process with a given period T.

As a first approach, the frequency $\omega_0 = 4.4$ rad/s was taken. We choice such a ω_0 because it is associated to a point A in the Nyquist plot of $Gp^T(z)$. Then both SRPID and MRPID were tuned for different K_P, always satisfying the gain margin requirement. The results are depicted in Fig. 2(a), where the greater and lower phase margin obtained with the MRPID ($\times$) and with the SRPID ($\cdot$) are shown. Thus an upper and lower bound for phase margins are obtained. It can also be seen that just for $T_e = 20$ ms the MRPID is able to fulfil the robustness while the MRPID allows to have 20 ms $\leq T_e \leq$ 100 ms.

In a second approach all the points in the third quadrant, instead of a single point A, belonging to the Nyquist plot of $Gp^T(z)$ were scanned. Again both SRPID and MRPID where tuned with several K_P, always satisfying the gain margin requirement. The results have been depicted in Fig. 2(b), where the symbols × and · have the same meaning as above. Again an upper and lower bound is found for both controllers. Thus, any point within the solid lines means a phase margin that is reachable for a certain error period Te using the MRPID controller. The same can be said for the SRPID and the dotted lines. Notice that the figure proves that the requirements can be satisfy but does not include the information about neither K_P nor ω_0. However since Fig. 2 was obtained with an iterative method such information can be saved and retrieved whenever.

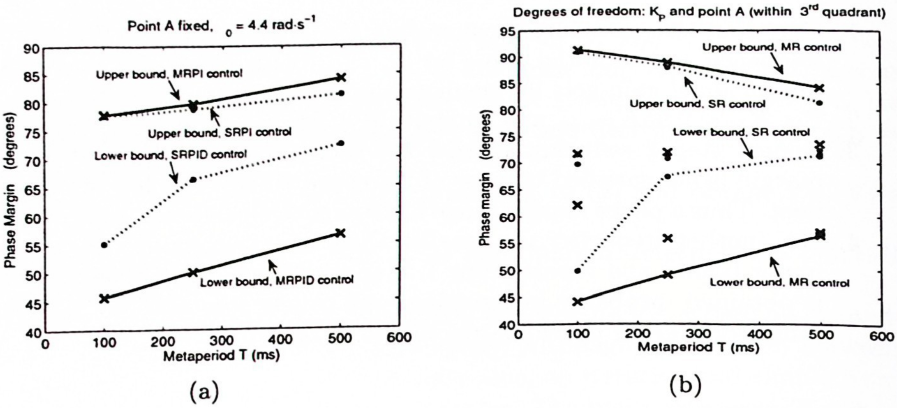
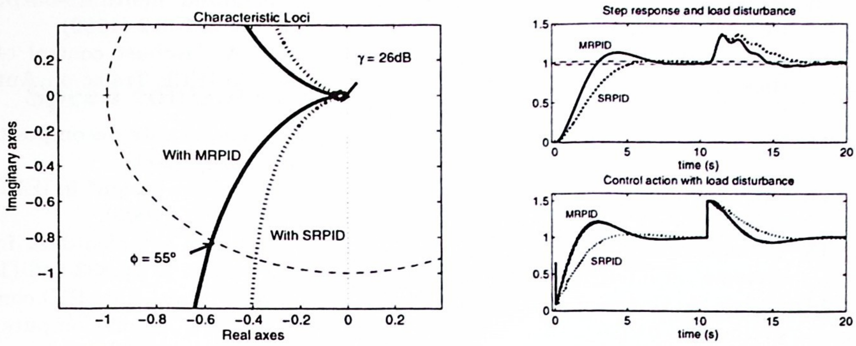


Fig. 2. Phase margin range when point A is fixed (a) and when is free within the 3^{rd} quadrant (b)

This example shows how MRPID controllers can improve the performance of a system in which there are severe sampling constrains in the sensors while the actuators can work much faster. For instance, from Fig. 2(b) it can be seen that if $T_e = 125$ ms and $T_u = 50$ ms then it is impossible to satisfy both gain and phase margin requirements using a conventional SRPID but it is using a MRPID instead. Thus, tuning the both SRPID and MRPID to obtain the phase margin closest to 55° satisfying the gain margin we obtain the characteristic loci, the step response and control action depicted in Fig. 3.

5 Conclusions

This paper introduces an indirect tuning method for the derivative gain for the more general multi-rate situation. It is indirect because decisions are taken over K_P in terms of robustness requirements and K_D is recomputed in terms of K_P. This is why MRPID controllers are an interesting control strategy, not only for coping with sampling constrains either in the error or in the control action, but also for taking advantage of the sampling period considering it as an extra parameter since SRPID can be considered as a particular case. Due to it



(a) (b)

Fig. 3. Characteristic loci (a), step response, load rejection and control action (b) with both MRPID and SRPID. The sampling constrains are $T_e = 125$ ms and $T_u = 50$ ms and 55° are required. The SRPID is not able to satisfy it.

is necessary to solve an eigenvalue equation it not possible to give an analytical solution and even for those cases in which it is, the computational solution is more straightforward.

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References

1. O'Dwyer, A. "Handbook of PI and PID" Ed. Imperial College Press (2003).
2. López, I., Dormido, S., and Morilla, F. "The sampling period as a control parameter" in 2nd Symp. on Intelligent Components and Instruments for Control Applications, Budapest (1994).
3. González, R. López, I., Morilla, F and Pastor, R. "SintoLab: The REPSOL-YPF PID tuning tool" in Control Engineering Practice, Pergamon. pp. 1469-1480 (2003).
4. Morant, F., Albertos, P., and Crespo, A. "Oxide composition control in a raw material mill" in IFAC Symp. on Microprocessor in Control. Istambul (1986).
5. Ohshima, M., Hashimoto I., Takeda, M., Yoneyama, T. and Goto, F. "Multirate multivariable model predictive control and its application to a semi-commercial polymerization reactor" in Proc. of the ACC, 2, pp.1576-1581 (1992).
6. Nemani, M., Tsao, T.C., and Hutchinson, S. "Multirate analysis and design of visual feedback servo control systems" in ASME Journal of Dynamic Systems, Measurement and Control. Vol 116, no. 1. pp.45-55 (1994).
7. Araki, M., Yamamoto, K. "Multivariable multirate sampled-data systems: State-space description, transfer characteristics, and Nyquist criterion" in IEEE Trans. on Automatic Control, Vol. AC-31, No. 2, pp. 145-154. (1986).
8. Araki, M., Hagiwara, T. "Pole assignment by multirate-sampled data output feedback" in Int. journal of control. Vo. 44, pp.1661-1673 (1986).

9. Hagiwara, T., Fujimura, T.F., Araki, M. "Generalizad multirate-output controllers" in Int. Journal of Control. Vol 52, No. 3, pp. 597-612 (1990).
10. Khargonekar, P.P., Poolla, K., and Tannenbaum, A. "Robust control of linear time invatiant plants using periodic compensation" in IEEE Trans. on Automatic Control, Vol. AC-30. No. 11, pp.1088-1096.
11. Salt, J. and Albertos, P. "Multirate controllers design by rate decomposition" in Proc. of 30th conference on decision and control, Sydney (2000).
12. Salt, J. and Albertos, P. "Model-based multirate controllers design" in IEEE trans. On control systems technology. Vol. 13, No.6, pp. 988-997 (2005).
13. Thompson, P.M. "Gain and phase margins of multirate sampled-data feedback systems" in International Journal of Control. Vol. 44, No. 3, pp.833-846 (1986).
14. Cuesta, A. Grau, L., López, I. "CACSD tools for tuning Multi-Rate PID controllers in time and frequency domain" in Proc. of the IEEE Int. Conf. on Computer Aided Control System Design. pp. 3036-3041. Munich, Germany (2006).
15. Kranc, G.M. "Input-output analysis of multirate feedback systems". IEEE Trans. on Automatic Control. Vol. AC-3, pp.21-28 (1957).
16. Anderson, B.D.O., Keller, J.P. "Discretization techniques in control systems". Control and Dynamic systems. Vol 66. pp. 47-93 (1994).
17. Cuesta, A. Grau, L., López, I. "Generalised tuning method for Multi-Rate PID controllers" in Proc. of the CSC'07, May 16-18, 2007, Marrakech, Morocco (2007).
18. Maceijowski, J.M. "Multivariable Feedback Design" Addison-Wesley. Cornwall, England (1989) Cuesta, A. Grau, L., López, I. "CACSD Tools for Tuning Multi-Rate PID Controllers in Time and Frequency Domains" in Proc. of the IEEE Symposium on CACSD, Munich (2006).